

ON THE ALGEBRAIC K-THEORY OF \mathbb{Z}

Dominique ARLETTAZ

Département de mathématiques, École Polytechnique Fédérale, CH-1015, Lausanne, Switzerland

Communicated by E.M. Friedlander

Received 24 April 1986

Let $GL(\mathbb{Z})$ (respectively $SL(\mathbb{Z})$) be the infinite general (respectively special) linear group and $St(\mathbb{Z})$ the infinite Steinberg group of \mathbb{Z} . This paper studies the relationships between $K_i\mathbb{Z} := \pi_i BGL(\mathbb{Z})^+$, $H_i(SL(\mathbb{Z}); \mathbb{Z})$ and $H_i(St(\mathbb{Z}); \mathbb{Z})$ for $i = 4$ and 5 (they are well understood for $i \leq 3$). The main results describe the Hurewicz homomorphism $K_i\mathbb{Z} \rightarrow H_i(St(\mathbb{Z}); \mathbb{Z})$: it is an isomorphism if $i = 4$ and its cokernel is cyclic of order 2 if $i = 5$ (more precisely, the induced homomorphism $K_5\mathbb{Z}/\text{torsion} \rightarrow H_5(St(\mathbb{Z}); \mathbb{Z})/\text{torsion}$ is multiplication by 2). The relations between the integral homology of $St(\mathbb{Z})$ and that of $SL(\mathbb{Z})$ in dimensions 4 and 5 are also explained.

Introduction

Let $GL(\mathbb{Z})$ be the infinite general linear group, $SL(\mathbb{Z})$ the infinite special linear group and $St(\mathbb{Z})$ the infinite Steinberg group of the ring of integers \mathbb{Z} . The relations between the groups $K_i\mathbb{Z} := \pi_i BGL(\mathbb{Z})^+$, $H_i(SL(\mathbb{Z}); \mathbb{Z})$ and $H_i(St(\mathbb{Z}); \mathbb{Z})$ are well understood for $i \leq 3$: $K_1\mathbb{Z} \cong \mathbb{Z}/2$, $H_1(SL(\mathbb{Z}); \mathbb{Z}) = H_1(St(\mathbb{Z}); \mathbb{Z}) = 0$; $K_2\mathbb{Z} \cong H_2(SL(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/2$, $H_2(St(\mathbb{Z}); \mathbb{Z}) = 0$; $K_3\mathbb{Z} \cong H_3(St(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/48$ and the Hurewicz homomorphism $K_3\mathbb{Z} \rightarrow H_3(SL(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$ is surjective [1, Satz 1.5]. The purpose of this paper is to study these relations for $i = 4$ and 5 .

1. Statement of the main results

In this section we present the main results of the paper.

Theorem 1.1. $H_4(SL(\mathbb{Z}); \mathbb{Z}) \cong H_4(St(\mathbb{Z}); \mathbb{Z}) \oplus \mathbb{Z}/2$.

Theorem 1.2. Let g denote the surjective canonical homomorphism $St(\mathbb{Z}) \twoheadrightarrow SL(\mathbb{Z})$ whose kernel is $K_2\mathbb{Z}$ and g_* the induced homomorphism $H_5(St(\mathbb{Z}); \mathbb{Z}) \rightarrow H_5(SL(\mathbb{Z}); \mathbb{Z})$. Then $H_5(SL(\mathbb{Z}); \mathbb{Z}) \cong (\text{Im } g_*) \oplus L$, where $L \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Now let Hu denote the Hurewicz homomorphism $K_*\mathbb{Z} \rightarrow H_*(St(\mathbb{Z}); \mathbb{Z})$.

Theorem 1.3. $Hu: K_4\mathbb{Z} \rightarrow H_4(\text{St}(\mathbb{Z}); \mathbb{Z})$ is an isomorphism.

Theorem 1.4. *There exists an exact sequence*

$$K_5\mathbb{Z} \xrightarrow{Hu} H_5(\text{St}(\mathbb{Z}); \mathbb{Z}) \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Theorem 1.5. *The homomorphisms*

$$K_5\mathbb{Z}/\text{torsion} \rightarrow H_5(\text{St}(\mathbb{Z}); \mathbb{Z})/\text{torsion}$$

and

$$K_5\mathbb{Z}/\text{torsion} \rightarrow H_5(\text{SL}(\mathbb{Z}); \mathbb{Z})/\text{torsion},$$

induced by the Hurewicz homomorphism, are multiplications by 2. (Recall that these three groups are infinite cyclic.)

We shall prove Theorems 1.1 and 1.2 in Section 2, Theorems 1.3, 1.4 and 1.5 in Section 4.

2. The group extension $K_2\mathbb{Z} \rightarrow \text{St}(\mathbb{Z}) \rightarrow \text{SL}(\mathbb{Z})$

In this section we study the relationships between the (co)homology of $\text{St}(\mathbb{Z})$ and that of $\text{SL}(\mathbb{Z})$ using the Serre spectral sequence of the universal central extension

$$K_2\mathbb{Z} \cong \mathbb{Z}/2 \rightarrow \text{St}(\mathbb{Z}) \xrightarrow{g} \text{SL}(\mathbb{Z}).$$

In particular we will prove Theorems 1.1 and 1.2.

We start by restricting our attention to cohomology with $\mathbb{Z}/2$ -coefficients. Let us first recall the following lemma (cf. [7, p. 154, Corollary 8.12]) which describes the structure of $H^*(\text{SL}(\mathbb{Z}); \mathbb{Z}/2)$, since the cohomology of the group $\text{SL}(\mathbb{Z})$ is the same as that of the H -space $B\text{SL}(\mathbb{Z})^+$.

Lemma 2.1. *Let X be a connected H -space of finite type, then $H^*(X; \mathbb{Z}/2) = \bigotimes_{i=0}^{\infty} B_i$, where each*

$$B_i = \begin{cases} \mathbb{Z}/2[x_i] & \text{or} \\ \mathbb{Z}/2[x_i]/(x_i^{e_i} = 0) & \text{where } e_i \text{ is a power of } 2. \end{cases} \quad \square$$

We want to look more precisely at the mod 2 cohomology classes of $\text{SL}(\mathbb{Z})$. It is well known that all Stiefel–Whitney classes w_i ($\deg w_i = i$) are non-zero (except $w_1 = 0$) and algebraically independent in $H^*(\text{SL}(\mathbb{Z}); \mathbb{Z}/2)$ [6]. Because the space $B\text{SL}(\mathbb{Z})^+$ is simply connected we have

$$H^1(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) = 0$$

and

$$H^2(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathrm{Hom}(K_2\mathbb{Z}, \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

generated by w_2 .

Corollary 2.2. $H^*(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) = \mathbb{Z}/2[w_2] \otimes (\bigotimes_{i=1}^{\infty} B_i)$, where each B_i is as in Lemma 2.1 and generated by one element of degree ≥ 3 . \square

We know that

$$H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

(since $H_3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$ [1, Satz 1.5]); one copy of $\mathbb{Z}/2$ is generated by w_3 and we define the generator α of the other copy as follows: let h be the homomorphism $\mathrm{SL}(\mathbb{Z}) \rightarrow \mathrm{SL}(\mathbb{F}_3)$ induced by the reduction mod 3 (\mathbb{F}_3 is the field of three elements) and $h^*: H^3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}/2) \cong \mathbb{Z}/2 \rightarrow H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$ the induced homomorphism which is injective because of the surjectivity of $h_*: H_3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24 \rightarrow H_3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}) \cong \mathbb{Z}/8$ (cf. [1, §3]).

Definition 2.3. Let us call α the image of the generator of $H^3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}/2)$ under the homomorphism h^* ($\alpha \in H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$).

We shall use the following notation: $\beta: H^*(-; \mathbb{Z}/2) \rightarrow H^{*+1}(-; \mathbb{Z})$ denotes the Bockstein homomorphism and $\mathrm{red}_2: H^*(-; \mathbb{Z}) \rightarrow H^*(-; \mathbb{Z}/2)$ the reduction mod 2 associated with the short exact sequence $\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2$; we also need the Steenrod square operations $\mathrm{Sq}^i: H^*(-; \mathbb{Z}/2) \rightarrow H^{*+i}(-; \mathbb{Z}/2)$ ($\mathrm{Sq}^1 = \mathrm{red}_2 \circ \beta$).

Remark 2.4. (a) We deduce from the definition of α and from the commutative diagram

$$\begin{array}{ccc} H^3(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}/2) & \xrightarrow{\beta} & H^4(\mathrm{SL}(\mathbb{F}_3); \mathbb{Z}) \\ \downarrow h^* & & \downarrow h^* \\ H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) & \xrightarrow{\beta} & H^4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \end{array}$$

that $\beta(\alpha) \neq 0$. On the other hand since $w_3 = \mathrm{Sq}^1 w_2$ (by Wu's formula) one has $\beta(w_3) = 0$. Consequently $\alpha \neq w_3$.

(b) $\mathrm{Sq}^1 \alpha = \mathrm{red}_2(\beta(\alpha)) = 0$ because $\beta(\alpha)$ is the element of order 2 in $H^4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z}/24$.

Definition 2.5. Let z be a generator of the infinite cyclic group $\mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z})$ and ζ an element of $H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ such that $\rho(\zeta) = z$, where ρ is the

homomorphism $H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \rightarrow \mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z})$ given by the universal coefficient theorem. Finally we define $\eta := \mathrm{red}_2(\zeta) \in H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$.

Remark 2.6. (a) $\mathrm{Sq}^1 \eta = 0$ since $\beta(\eta) = 0$.

(b) Let ρ denote now the homomorphism $H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \rightarrow \mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z}/2)$, then $\rho(\eta) \neq 0$ (because $\rho(\eta) = \mathrm{red}_2(z) \neq 0$).

Lemma 2.7. *The cohomology classes w_5 , $w_2 w_3$, $w_2 \alpha$ and η are linearly independent in $H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$.*

Proof. Let $r, s, t, u \in \{0, 1\}$ such that $rw_5 + sw_2 w_3 + tw_2 \alpha + u\eta = 0$. Using Wu's formula we obtain $\mathrm{Sq}^1(rw_5 + sw_2 w_3 + tw_2 \alpha + u\eta) = sw_3^2 + tw_3 \alpha = 0$ and we may conclude, according to Lemma 2.1, that $s = t = 0$. We apply the homomorphism ρ to the remaining equation $rw_5 + u\eta = 0$: since $w_5 = \mathrm{Sq}^1 w_4$ we get $\rho(w_5) = 0$ and $u\rho(\eta) = 0$; Remark 2.6(b) then implies that $u = 0$ and $r = 0$. \square

We are now able to work with the Serre spectral sequence of $\mathbb{Z}/2 \hookrightarrow \mathrm{St}(\mathbb{Z}) \xrightarrow{g} \mathrm{SL}(\mathbb{Z})$, whose E_2 -term is $H^*(\mathrm{SL}(\mathbb{Z}); H^*(\mathbb{Z}/2; \mathbb{Z}/2)) \cong H^*(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \otimes \mathbb{Z}/2[x]$ (x denotes the generator of $H^*(\mathbb{Z}/2; \mathbb{Z}/2) = \mathbb{Z}/2[x]$, $\deg x = 1$); in particular $E_2^{1,j} = 0 \ \forall j \geq 0$. Because $H^1(\mathrm{St}(\mathbb{Z}); \mathbb{Z}/2) = 0$ we get $d_2(x) = w_2$ and, $\forall n \geq 0$ and $\forall y \in H^*(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$, $d_2(x^{2n}y) = 0$, $d_2(x^{2n+1}y) = x^{2n}w_2 y$; it follows from Corollary 2.2 that $d_2(x^{2n+1}y) \neq 0$ if $y \neq 0$. Consequently the E_3 -term has the following properties: $E_3^{1,j} = E_3^{2,j} = 0 \ \forall j \geq 0$, $E_3^{i,2n+1} = 0 \ \forall i, n \geq 0$, $E_3^{3,2n} = E_2^{3,2n} \ \forall n \geq 0$. In order to understand the action of d_3 we use the fact that the Sq^i operations commute with the transgression: $d_3(x^2) = d_3(\mathrm{Sq}^1 x) = \mathrm{Sq}^1(d_2(x)) = \mathrm{Sq}^1 w_2 = w_3$ and, $\forall y \in E_3^{*,0}$, $d_3(x^2 y) = w_3 y$; again the structure of $H^*(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)$ implies that $d_3(x^2 y) \neq 0$ if $y \neq 0$. Obviously $d_3(x^4) = 0$, $d_4(x^4) = 0$ but, as above, $d_5(x^4) = d_5(\mathrm{Sq}^2 x^2) = \mathrm{Sq}^2(d_3(x^2)) = \mathrm{Sq}^2 w_3$; by Wu's formula $\mathrm{Sq}^2 w_3 = w_5 + w_2 w_3$ and we conclude that $d_5(x^4) = w_5$ since $w_2 w_3 = 0$ in $E_3^{5,0}$.

We summarize the information we have on $E_\infty^{i,j}$ for $i + j \leq 5$: $E_\infty^{i,j} = 0$ for $i + j \leq 5$, $j > 0$; $E_\infty^{1,0} = E_\infty^{2,0} = 0$; $E_\infty^{3,0} \cong H^3(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)/(w_3 = 0)$; $E_\infty^{4,0} \cong H^4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)/(w_2^2 = 0)$; $E_\infty^{5,0} \cong H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)/(w_5 = w_2 w_3 = w_2 \alpha = 0)$. This implies the following result:

Lemma 2.8. *The homomorphism $g: \mathrm{St}(\mathbb{Z}) \rightarrow \mathrm{SL}(\mathbb{Z})$ induces a surjective homomorphism $g^*: H^i(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2) \rightarrow H^i(\mathrm{St}(\mathbb{Z}); \mathbb{Z}/2)$ for $i \leq 5$.*

For $i = 2$ $\ker g^* \cong \mathbb{Z}/2$, generated by w_2 ; for $i = 3$ $\ker g^* \cong \mathbb{Z}/2$, generated by w_3 ; for $i = 4$ $\ker g^* \cong \mathbb{Z}/2$, generated by w_2^2 ; for $i = 5$ $\ker g^* \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by w_5 , $w_2 w_3$ and $w_2 \alpha$. \square

For the next step of our argument we look at the Serre spectral sequence for integral cohomology of the central extension $\mathbb{Z}/2 \hookrightarrow \mathrm{St}(\mathbb{Z}) \xrightarrow{g} \mathrm{SL}(\mathbb{Z})$. Since $E_2^{i,j} \cong H^j(\mathrm{SL}(\mathbb{Z}); H^i(\mathbb{Z}/2; \mathbb{Z}))$ fulfils $E_2^{i,j} = 0$ for j odd, we have $E_2 = E_3$.

Lemma 2.9. $H^5(\text{St}(\mathbb{Z}); \mathbb{Z}) \cong H^5(\text{SL}(\mathbb{Z}); \mathbb{Z})/(\mathbb{Z}/2)$.

Proof. The unique non-zero terms $E_3^{i,j}$ of the line $i+j=5$ are $E_3^{3,2} \cong H^3(\text{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $E_3^{5,0} \cong H^5(\text{SL}(\mathbb{Z}); \mathbb{Z})$. We can show that the differential $d_3: E_3^{3,2} \rightarrow E_3^{5,0}$ is injective by reducing the spectral sequence mod 2 and looking at the corresponding differential of the mod 2 spectral sequence, which is injective; consequently $E_\infty^{3,2} = 0$. For the same reason $d_5: E_5^{0,4} = E_2^{0,4} \cong \mathbb{Z}/2 \rightarrow E_5^{5,0}$ is injective and $E_\infty^{5,0} \cong E_5^{5,0}/(\mathbb{Z}/2)$. It is possible to deduce from $|H^4(\text{St}(\mathbb{Z}); \mathbb{Z})| = 2|E_\infty^{4,0}|$ ($|\cdot|$ denotes the order of the group) that $E_5^{5,0} = E_2^{5,0}$; therefore $E_\infty^{5,0} \cong H^5(\text{SL}(\mathbb{Z}); \mathbb{Z})/(\mathbb{Z}/2)$ and the lemma is proved. \square

We finally consider the Serre spectral sequence for integral homology of the central extension $\mathbb{Z}/2 \hookrightarrow \text{St}(\mathbb{Z}) \xrightarrow{g} \text{SL}(\mathbb{Z})$: $E_{i,j}^2 \cong H_i(\text{SL}(\mathbb{Z}); H_j(\mathbb{Z}/2; \mathbb{Z}))$ satisfies $E_{i,j}^2 = 0$ for $i=1$ or j even, $j \geq 2$. We will need the following information:

Lemma 2.10. The differential $d^3: E_{3,1}^3 \rightarrow E_{0,3}^3 \cong \mathbb{Z}/2$ is zero.

Proof. Suppose $d^3: E_{3,1}^3 \rightarrow E_{0,3}^3$ is surjective. Then $E_{0,3}^\infty = 0$ and, because $E_{1,2}^\infty = 0$, $|H_3(\text{St}(\mathbb{Z}); \mathbb{Z})| = 2|E_{3,0}^\infty|$, we must have $E_{2,1}^\infty \cong \mathbb{Z}/2$ which implies that $d^2: E_{4,0}^2 \rightarrow E_{2,1}^2 \cong \mathbb{Z}/2$ is zero and that $E_{4,0}^\infty = E_{4,0}^2 \cong H_4(\text{SL}(\mathbb{Z}); \mathbb{Z})$. Thus the induced homomorphism $g_*: H_4(\text{St}(\mathbb{Z}); \mathbb{Z}) \rightarrow H_4(\text{SL}(\mathbb{Z}); \mathbb{Z})$ is surjective.

On the other hand we consider the following commutative diagram:

$$\begin{array}{ccccccc} K_4\mathbb{Z} & \xrightarrow{\text{Hu}} & H_4(B\text{St}(\mathbb{Z})^+; \mathbb{Z}) & & & & \\ \parallel & & \downarrow g_* & & & & \\ K_4\mathbb{Z} & \xrightarrow{\text{Hu}} & H_4(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}) & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & K_3\mathbb{Z} \xrightarrow{\text{Hu}} H_3(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}) \end{array}$$

where Hu denotes the Hurewicz homomorphism. The bottom sequence is the Whitehead exact sequence of $B\text{SL}(\mathbb{Z})^+$ and the kernel of $\text{Hu}: K_3\mathbb{Z} \rightarrow H_3(B\text{SL}(\mathbb{Z})^+; \mathbb{Z})$ is cyclic of order 2 (cf. [8] and [1, Satz 1.5]). Since $B\text{St}(\mathbb{Z})^+$ is 2-connected, $\text{Hu}: K_4\mathbb{Z} \rightarrow H_4(B\text{St}(\mathbb{Z})^+; \mathbb{Z})$ is surjective and g_* cannot be surjective: that gives us a contradiction. \square

Lemma 2.11. There exist the following exact sequences:

- (a) $0 \rightarrow H_4(\text{St}(\mathbb{Z}); \mathbb{Z}) \xrightarrow{g_*} H_4(\text{SL}(\mathbb{Z}); \mathbb{Z}) \rightarrow \mathbb{Z}/2 \rightarrow 0$,
- (b) $H_5(\text{St}(\mathbb{Z}); \mathbb{Z}) \xrightarrow{g_*} H_5(\text{SL}(\mathbb{Z}); \mathbb{Z}) \rightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow 0$.

Proof. It follows from the previous lemma that $E_{3,1}^\infty = E_{3,1}^3$ and, since $|H_3(\text{St}(\mathbb{Z}); \mathbb{Z})| = 2|E_{3,0}^\infty|$, that $E_{4,0}^\infty$ is the kernel of a surjective homomorphism $H_4(\text{SL}(\mathbb{Z}); \mathbb{Z}) \twoheadrightarrow \mathbb{Z}/2$. Obviously $E_{5,0}^\infty = E_{5,0}^3 = \ker d^2: E_{5,0}^2 \rightarrow E_{3,1}^2$. (Note that $E_{5,0}^2 \cong H_5(\text{SL}(\mathbb{Z}); \mathbb{Z})$ and $E_{3,1}^2 \cong H_3(\text{SL}(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.) We obtain the

exact sequence

$$\begin{aligned} H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) &\xrightarrow{g_*} E_{5,0}^2 \xrightarrow{d^2} E_{3,1}^2 \rightarrow H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \\ &\xrightarrow{g_*} H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \rightarrow \mathbb{Z}/2 \rightarrow 0. \end{aligned}$$

We deduce from the universal coefficient theorem and Borel's theorem that $H^5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ext}(H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z})$ and $H^5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{Ext}(H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z})$. Lemma 2.9 then implies the injectivity of $g_*: H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \rightarrow H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$. \square

Proof of Theorem 1.1. Since we know from Lemma 2.8 that $H^4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}/2) \cong H^4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}/2)/(\mathbb{Z}/2)$, the universal coefficient theorem gives us: $\mathrm{Hom}(H_4(\mathrm{St}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z}/2) \cong \mathrm{Hom}(H_4(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z}/2)/(\mathbb{Z}/2)$. Therefore we may conclude that the short exact sequence of Lemma 2.11(a) splits. \square

Proof of Theorem 1.2. It follows again from Lemma 2.8, the universal coefficient theorem and Theorem 1.1 that $\mathrm{Hom}(H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z}/2) \cong \mathrm{Hom}(H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z}), \mathbb{Z}/2)/(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$. The assertion of Theorem 1.2 is then a consequence of Lemma 2.11(b). \square

Remark 2.12. Recall that by Borel's theorem $H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z})/\mathrm{torsion}$ and $H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})/\mathrm{torsion}$ are infinite cyclic groups. According to Theorem 1.2, $g_*: H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \rightarrow H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ induces an isomorphism

$$H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z})/\mathrm{torsion} \xrightarrow{\cong} H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})/\mathrm{torsion}.$$

Remark 2.13. It is possible to show that the order of the kernel of $g_*: H_5(\mathrm{St}(\mathbb{Z}); \mathbb{Z}) \rightarrow H_5(\mathrm{SL}(\mathbb{Z}); \mathbb{Z})$ divides 4.

3. Homological relations between the spaces $B\mathrm{St}(\mathbb{Z})^+$ and $B\mathrm{SL}(\mathbb{F}_3)^+$

We consider now the commutative triangle

$$\begin{array}{ccc} B\mathrm{St}(\mathbb{Z})^+ & \xrightarrow{f} & B\mathrm{SL}(\mathbb{F}_3)^+ \\ & \searrow g & \nearrow h \\ & B\mathrm{SL}(\mathbb{Z})^+ & \end{array}$$

where f and h are induced by the reduction mod 3 and g is the map induced by the canonical homomorphism of Section 2 (recall that $\mathrm{St}(\mathbb{F}_3) = \mathrm{SL}(\mathbb{F}_3)$ and that $B\mathrm{SL}(\mathbb{F}_3)^+$ is 2-connected since $K_2\mathbb{F}_3 = 0$). In order to prove our main results

in Section 4 we need to examine the image of $f_*: H_5(BSt(\mathbb{Z})^+; \mathbb{Z}) \rightarrow H_5(BSL(\mathbb{F}_3)^+; \mathbb{Z})$.

We shall use throughout this section the following notation. We define F (respectively \bar{F}) as the fibre of h (respectively f) and get the commutative diagram

$$\begin{array}{ccccc} \bar{F} & \xrightarrow{i} & BSt(\mathbb{Z})^+ & \xrightarrow{f} & BSL(\mathbb{F}_3)^+ \\ \downarrow k & & \downarrow g & & \parallel \\ F & \xrightarrow{i} & BSL(\mathbb{Z})^+ & \xrightarrow{h} & BSL(\mathbb{F}_3)^+ \end{array}$$

where both rows are fibrations. As usual we shall denote by $j^*, f^*, i^*, h^*, k^*, g^*$ the induced homomorphisms in cohomology. If y is an element of $H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$ let us define $\tilde{y} := i^*(y) \in H^*(F; \mathbb{Z}/2)$ and $\bar{y} := g^*(y) \in H^*(BSt(\mathbb{Z})^+; \mathbb{Z}/2)$. According to [5] the ring $H^*(BSL(\mathbb{F}_3)^+; \mathbb{Z}/2)$ is generated by cohomology classes e_i and c_i , $i \geq 2$, where $\deg e_i = 2i - 1$ and $\deg c_i = 2i$.

Remark 3.1. (a) The space F is simply connected and \bar{F} is 2-connected. The groups $H_i(F; \mathbb{Z})$ are finite for $i = 2, 3, 4$ and $H_5(F; \mathbb{Z}) \cong \mathbb{Z} \oplus (\text{finite group})$, because the same results hold for $BSL(\mathbb{Z})^+$ and all homology groups of $BSL(\mathbb{F}_3)^+$ are finite (this is also true for \bar{F}). Note that a Serre spectral sequence argument shows that $H_2(F; \mathbb{Z}) \cong \mathbb{Z}/2$ and $H_3(F; \mathbb{Z}) \cong \mathbb{Z}/3$.

(b) The relation between F and the classifying space of the congruence subgroup of $SL(\mathbb{Z})$ of level 3 is explained in [2, Section 1].

We start by looking at mod 2 cohomology. Obviously $\tilde{w}_2 \neq 0$ in $H^2(F; \mathbb{Z}/2)$ and, since $h^*(e_2) = \alpha \in H^3(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$ (cf. Definition 2.3), $\tilde{\alpha} = 0$ but $\tilde{w}_3 \neq 0$ in $H^3(F; \mathbb{Z}/2)$. Because h is an H -map, F is an H -space and, by Lemma 2.1, $\tilde{w}_2 \tilde{w}_3 \neq 0$ in $H^5(F; \mathbb{Z}/2)$.

Lemma 3.2. *Let γ_m denote the homomorphism $H^*(F; \mathbb{Z}/2) \rightarrow H^*(F; \mathbb{Z}/2^m)$ induced by the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2^m$. Then $\gamma_m(\tilde{w}_2 \tilde{w}_3) \neq 0$ for all $m \geq 1$.*

Proof. Let $m \geq 1$ be a given integer and θ_m the homomorphism $H^*(F; \mathbb{Z}/2^m) \rightarrow H^*(F; \mathbb{Z}/2)$ induced by the surjection $\mathbb{Z}/2^m \rightarrow \mathbb{Z}/2$. We call a (respectively b) the generator of $H^2(F; \mathbb{Z}/2^m) \cong \text{Hom}(H_2(F; \mathbb{Z}), \mathbb{Z}/2^m) \cong \mathbb{Z}/2$ (respectively of $H^3(F; \mathbb{Z}/2^m) \cong \text{Ext}(H_2(F; \mathbb{Z}), \mathbb{Z}/2^m) \cong \mathbb{Z}/2$). It is clear that $\theta_m(b) = \tilde{w}_3$ which implies actually the equality $\tilde{w}_2 b = \tilde{w}_2 \tilde{w}_3$ in $H^5(F; \mathbb{Z}/2)$. On the other hand we deduce from $\gamma_m(\tilde{w}_2) = a$ that $\gamma_m(\tilde{w}_2 b) = ab$; therefore $\gamma_m(\tilde{w}_2 \tilde{w}_3) = ab$.

We complete the proof by showing that $ab \neq 0$ in $H^5(F; \mathbb{Z}/2^m)$. Let $\mu^*: H^*(F; \mathbb{Z}/2^m) \rightarrow H^*(F \times F; \mathbb{Z}/2^m)$ denote the homomorphism induced by the H -space structure of F . Since F is simply connected we have obviously $\mu^*(a) = a \otimes 1 + 1 \otimes a$ and $\mu^*(b) = b \otimes 1 + 1 \otimes b$. If $ab = 0$, then $0 = \mu^*(ab) = \mu^*(a)\mu^*(b) = a \otimes b + b \otimes a$ in $H^5(F \times F; \mathbb{Z}/2^m)$, which is not the case. \square

Corollary 3.3. *Let $\rho : H^5(F; \mathbb{Z}/2) \rightarrow \text{Hom}(H_5(F; \mathbb{Z}), \mathbb{Z}/2)$ be the homomorphism given by the universal coefficient theorem. Then $\rho(\tilde{w}_2 \tilde{w}_3) \neq 0$.*

Proof. Suppose $\rho(\tilde{w}_2 \tilde{w}_3) = 0$; then the exactness of the sequence $\text{Ext}(H_4(F; \mathbb{Z}), \mathbb{Z}/2) \xrightarrow{\nu} H^5(F; \mathbb{Z}/2) \xrightarrow{\rho} \text{Hom}(H_5(F; \mathbb{Z}), \mathbb{Z}/2)$ implies the existence of an element $\sigma \in \text{Ext}(H_4(F; \mathbb{Z}), \mathbb{Z}/2)$ such that $\nu(\sigma) = \tilde{w}_2 \tilde{w}_3$. Let 2^{m-1} be the exponent of the 2-torsion subgroup of $H_4(F; \mathbb{Z})$ and let us consider the commutative diagram

$$\begin{array}{ccc} \text{Ext}(H_4(F; \mathbb{Z}), \mathbb{Z}/2) & \xrightarrow{\gamma'_m} & \text{Ext}(H_4(F; \mathbb{Z}), \mathbb{Z}/2^m) \\ \downarrow \nu & & \downarrow \\ H^5(F; \mathbb{Z}/2) & \xrightarrow{\gamma_m} & H^5(F; \mathbb{Z}/2^m) \end{array}$$

where γ_m and γ'_m are induced by the inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/2^m$. It follows from the Hom-Ext-sequence that $\gamma'_m = 0$; therefore $\gamma_m(\tilde{w}_2 \tilde{w}_3) = \gamma_m(\nu(\sigma)) = 0$, which contradicts the previous lemma. \square

Lemma 3.4. *The element $\rho(\tilde{w}_2 \tilde{w}_3)$ belongs to the image of the reduction mod 2 $\text{Hom}(H_5(F; \mathbb{Z}), \mathbb{Z}) \rightarrow \text{Hom}(H_5(F; \mathbb{Z}), \mathbb{Z}/2)$.*

Proof. We look at the commutative diagram

$$\begin{array}{ccc} H^5(BSL(\mathbb{Z})^+; \mathbb{Z}/2) & \xrightarrow{\beta} & H^6(BSL(\mathbb{Z})^+; \mathbb{Z}) \\ \downarrow i^* & & \downarrow i^* \\ H^5(F; \mathbb{Z}/2) & \xrightarrow{\beta} & H^6(F; \mathbb{Z}) \end{array}$$

where β denotes again the Bockstein homomorphism. It is easy to check that $\beta(w_2 w_3) = c_3(\text{SL}(\mathbb{Z}))$, i.e., the third Chern class of the inclusion $\text{SL}(\mathbb{Z}) \hookrightarrow \text{GL}(\mathbb{C})$ (cf. [1]), because this equality holds in the cohomology of BSO . We then deduce from $i^*(c_3(\text{SL}(\mathbb{Z}))) = 0$ [3] that $\beta(\tilde{w}_2 \tilde{w}_3) = \beta(i^*(w_2 w_3)) = i^*(\beta(w_2 w_3)) = 0$. Consequently $\tilde{w}_2 \tilde{w}_3$ belongs to the image of the reduction mod 2 and the same is true for $\rho(\tilde{w}_2 \tilde{w}_3)$. \square

Lemma 3.5. *$k^*(\tilde{w}_2 \tilde{w}_3) = 0$ in $H^5(\bar{F}; \mathbb{Z}/2)$.*

Proof. This follows from $g^*(w_2 w_3) = 0$ (cf. Lemma 2.8) since $k^*(\tilde{w}_2 \tilde{w}_3) = j^*(g^*(w_2 w_3))$. \square

We are now able to prove the main result of this section. Recall that $H_5(BSL(\mathbb{F}_3)^+; \mathbb{Z}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/13$ [4].

Proposition 3.6. *The 2-torsion subgroup of the image of the homomorphism $f_* : H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z}) \rightarrow H_5(BSL(\mathbb{F}_3)^+; \mathbb{Z})$ is cyclic of order 2.*

Proof. The homomorphisms g, i, j, k induce the commutative diagram

$$\begin{array}{ccc} H_5(\bar{F}; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} & \xrightarrow{j_*} & H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \\ \downarrow k_* & & \downarrow g_* \\ H_5(F; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} & \xrightarrow{i_*} & H_5(B\text{SL}(\mathbb{Z})^+; \mathbb{Z})/\text{torsion} \cong \mathbb{Z} \end{array}$$

and we know that g_* is an isomorphism (Remark 2.12). It follows from Corollary 3.3. and Lemmas 3.4 and 3.5 that k_* is multiplication by an even number (or 0); thus j_* is also multiplication by an even number (or 0).

On the other hand the Serre spectral sequence of the fibration $\bar{F} \xrightarrow{j} B\text{St}(\mathbb{Z})^+ \xrightarrow{f} B\text{SL}(\mathbb{F}_3)^+$ produces an exact sequence $H_5(\bar{F}; \mathbb{Z}) \xrightarrow{j_*} H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z}) \xrightarrow{f_*} H_5(B\text{SL}(\mathbb{F}_3)^+; \mathbb{Z})$. Therefore the 2-torsion subgroup of the cokernel of j_* is cyclic of order 2 and the proof is complete. \square

Our next objective is to examine the image of the torsion subgroup of $H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z})$ under the homomorphism f_* . We first consider the homomorphism $h^*: H^*(B\text{SL}(\mathbb{F}_3)^+; \mathbb{Z}/2) \rightarrow H^*(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}/2)$; recall that by Definition 2.3 $h^*(e_2) = \alpha \in H^3(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}/2)$.

Lemma 3.7. $h^*(c_2) = w_2^2$, $h^*(e_3) = \text{Sq}^2\alpha$, $h^*(c_3) = w_3^2$.

Proof. Since $e_2^2 = c_3$ [5, p. 565] (and consequently $\text{Sq}^2 e_2 = e_3$) we get $h^*(c_3) = \alpha^2$ and $h^*(e_3) = \text{Sq}^2\alpha$.

We use the Eilenberg–Moore spectral sequence of the fibration $F \xrightarrow{i} B\text{SL}(\mathbb{Z})^+ \xrightarrow{h} B\text{SL}(\mathbb{F}_3)^+$ which converges to $H^*(F; \mathbb{Z}/2)$. Let R be the polynomial ring $H^*(B\text{SL}(\mathbb{F}_3)^+; \mathbb{Z}/2)$. In order to get the E_1 -term of this (second quadrant) spectral sequence we choose an R -free resolution of the field of two elements \mathbb{F}_2 :

$$\cdots \rightarrow \bigoplus_{k=1}^{\infty} Ru_k \rightarrow R \rightarrow \mathbb{F}_2$$

where $u_k^2 = 0 \ \forall k \geq 1$ and $\text{bideg } u_1 = (-1, 3)$, $\text{bideg } u_2 = (-1, 4)$, $\text{bideg } u_3 = (-1, 5)$, $\text{bideg } u_4 = (-1, 7), \dots$. We obtain E_1 by tensoring this resolution with $H^*(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}/2)$ over R ; in particular $E_1^{0,*} \cong H^*(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}/2)$.

We know that $i^*(\alpha) = 0$ and we deduce from [3] that $i^*(w_2^2) = 0$, $i^*(w_3^2) = 0$ (because w_2^2 (respectively w_3^2) is the reduction mod 2 of the second (respectively the third) Chern class of the inclusion $\text{SL}(\mathbb{Z}) \hookrightarrow \text{GL}(\mathbb{C})$). This implies that $\alpha \in E_1^{0,3}$, $w_2^2 \in E_1^{0,4}$ and $w_3^2 \in E_1^{0,6}$ have to be killed by some differential: for placement reasons these three classes belong to the image of the differential d_1 of bidegree $(1, 0)$. Since u_1 and u_2 generate $E_1^{-1,3}$ and $E_1^{-1,4}$ respectively, we have $d_1(u_1) = \alpha$ and $d_1(u_2) = w_2^2$ (that gives us $h^*(c_2) = w_2^2$). $E_1^{-1,6}$ is generated by

$w_3 u_1$, αu_1 , $w_2 u_2$ and therefore $\text{Im } d_1: E_1^{-1,6} \rightarrow E_1^{0,6}$ is generated by $w_3 \alpha$, α^2 and w_2^3 . We then may conclude that $w_3^2 = r w_3 \alpha + s \alpha^2 + t w_2^3$ for some $r, s, t \in \{0, 1\}$. But it follows from Lemma 2.1 that $r = t = 0$, $s = 1$: $w_3^2 = \alpha^2$; consequently $h^*(c_3) = w_3^2$. \square

Definition 3.8. $\xi := w_3 + \alpha \in H^3(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}/2)$.

Remark 3.9. (a) $\xi^2 = 0$ since $w_3^2 = \alpha^2$.

(b) $h^*(e_3) = \text{Sq}^2 \alpha = \text{Sq}^2(w_3 + \xi) = w_2 w_3 + w_5 + \text{Sq}^2 \xi$ by Wu's formula.

(c) Since $f^* = g^* \circ h^*$ it follows from Lemmas 2.8 and 3.7 that $f^*(e_2) = \bar{\alpha} = \bar{\xi}$ ($\neq 0$) generates $H^3(\text{BSt}(\mathbb{Z})^+; \mathbb{Z}/2)$ and that $f^*(c_2) = 0$, $f^*(e_3) = \text{Sq}^2 \bar{\xi}$, $f^*(c_3) = 0$.

Lemma 3.10. Let us call \hat{c}_3 a generator of $H^6(\text{BSL}(\mathbb{F}_3)^+; \mathbb{Z}) \cong \mathbb{Z}/26$, then $h^*(\hat{c}_3) = c_3(\text{SL}(\mathbb{Z})) \in H^6(\text{BSL}(\mathbb{Z})^+; \mathbb{Z})$ ($c_3(\text{SL}(\mathbb{Z}))$ is the third Chern class of the inclusion $\text{SL}(\mathbb{Z}) \hookrightarrow \text{GL}(\mathbb{C})$).

Proof. Let β denote again the Bockstein homomorphism $H^*(-; \mathbb{Z}/2) \rightarrow H^{*+1}(-; \mathbb{Z})$. The element $w_2 \alpha$ of $H^5(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}/2)$ satisfies $i^*(w_2 \alpha) = 0$ since $i^*(\alpha) = 0$. We define $\tau := \beta(w_2 \alpha)$; of course $i^*(\tau) = 0$ in $H^6(F; \mathbb{Z})$. Moreover we know from [3] that $i^*(c_3(\text{SL}(\mathbb{Z}))) = 0$; note that $\tau \neq c_3(\text{SL}(\mathbb{Z}))$ because $\text{red}_2(\tau) = \text{Sq}^1(w_2 \alpha) = w_3 \alpha \neq w_3^2 = \text{red}_2(c_3(\text{SL}(\mathbb{Z})))$. We can conclude by looking at the Serre spectral sequence of $F \xrightarrow{i} \text{BSL}(\mathbb{Z})^+ \xrightarrow{h} \text{BSL}(\mathbb{F}_3)^+$ that the kernel of $i^*: H^6(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}) \rightarrow H^6(F; \mathbb{Z})$ is generated by τ and $c_3(\text{SL}(\mathbb{Z}))$. But $h^*(\hat{c}_3)$ belongs to this kernel, i.e., $h^*(\hat{c}_3) = r\tau + sc_3(\text{SL}(\mathbb{Z}))$ where $r, s \in \{0, 1\}$. We apply red_2 to this equation and obtain $h^*(c_3) = r w_3 \alpha + s w_3^2$. On the other hand, since $h^*(c_3) = w_3^2$ by Lemma 3.7, we get $r = 0$, $s = 1$, so $h^*(\hat{c}_3) = c_3(\text{SL}(\mathbb{Z}))$. \square

Corollary 3.11. Let $\rho: H^5(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}/2) \twoheadrightarrow \text{Hom}(H_5(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}/2)$ be the homomorphism given by the universal coefficient theorem and η the cohomology class introduced in Definition 2.5. Then $\rho(\text{Sq}^2 \xi) = \rho(\eta)$.

Proof. The commutative diagram

$$\begin{array}{ccc} H^5(\text{BSL}(\mathbb{F}_3)^+; \mathbb{Z}/2) & \xrightarrow{f^*} & H^5(\text{BSt}(\mathbb{Z})^+; \mathbb{Z}/2) \\ \rho \downarrow \cong & & \downarrow \rho \\ \text{Hom}(H_5(\text{BSL}(\mathbb{F}_3)^+; \mathbb{Z}), \mathbb{Z}/2) & \xrightarrow{f^{\sqcup}} & \text{Hom}(H_5(\text{BSt}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}/2) \end{array}$$

where f^{\sqcup} is induced by f_* , and the injectivity of f^{\sqcup} (consequence of Proposition 3.6) give us: $\rho(\text{Sq}^2 \bar{\xi}) = \rho(f^*(e_3)) = f^{\sqcup}(\rho(e_3)) \neq 0$; since $\text{Sq}^2 \bar{\xi} = g^*(\text{Sq}^2 \xi)$ we get $\rho(\text{Sq}^2 \xi) \neq 0$ in $\text{Hom}(H_5(\text{BSL}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}/2)$.

It follows from Lemma 3.10 and from $\beta(e_3) = 13\hat{c}_3$ that $\beta(h^*(e_3)) = h^*(\beta(e_3)) = 13c_3(\text{SL}(\mathbb{Z})) = c_3(\text{SL}(\mathbb{Z}))$ because $c_3(\text{SL}(\mathbb{Z}))$ is an element of order 2

[1]. On the other hand, according to Remark 3.9(b), $\beta(h^*(e_3)) = \beta(w_2w_3) + \beta(w_5) + \beta(\text{Sq}^2\xi) = c_3(\text{SL}(\mathbb{Z})) + \beta(\text{Sq}^2\xi)$ ($\beta(w_5) = 0$ since $w_5 = \text{Sq}^1w_4$). Thus we get $\beta(\text{Sq}^2\xi) = 0$; therefore $\rho(\text{Sq}^2\xi)$ belongs to the image of the reduction mod 2 $\text{Hom}(H_5(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}) \rightarrow \text{Hom}(H_5(B\text{SL}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}/2)$. We then deduce from Definition 2.5 that $\rho(\text{Sq}^2\xi) = \rho(\eta)$. \square

Remark 3.12. The previous corollary implies that $\rho(\text{Sq}^2\bar{\xi}) = \rho(\bar{\eta})$ in $\text{Hom}(H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}/2)$.

We consider again the homomorphism $f_*: H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z}) \rightarrow H_5(B\text{SL}(\mathbb{F}_3)^+; \mathbb{Z})$.

Proposition 3.13. *Let T denote the torsion subgroup of $H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z})$. Then the 2-torsion subgroup of $f_*(T)$ is trivial.*

Proof. Because $\rho(\bar{\eta})$ is by definition an element of the image of the reduction mod 2 $\text{Hom}(H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}) \rightarrow \text{Hom}(H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z}), \mathbb{Z}/2)$, one has $\rho(\bar{\eta})(T) = 0$. Observe that, by commutativity of the diagram introduced in the proof of Corollary 3.11, $f^\square(\rho(e_3)) = \rho(\text{Sq}^2\bar{\xi}) = \rho(\bar{\eta})$. Consequently $f^\square(\rho(e_3))(T) = 0$ and, since f^\square is induced by f_* , $\rho(e_3)(f_*(T)) = 0$, which implies that there is no 2-torsion in $f_*(T)$. \square

4. The Whitehead sequence of the space $B\text{St}(\mathbb{Z})^+$

Proof of Theorems 1.3 and 1.4. We use the map $f: B\text{St}(\mathbb{Z})^+ \rightarrow B\text{SL}(\mathbb{F}_3)^+$ in order to compare the Whitehead exact sequence (cf. [7, p. 555, Theorem 3.12]) of $B\text{St}(\mathbb{Z})^+$ with that of $B\text{SL}(\mathbb{F}_3)^+$ (both spaces are 2-connected). We get the following commutative diagram where both rows are exact (Hu denotes the Hurewicz homomorphism):

$$\begin{array}{ccccccc}
 K_5\mathbb{Z} & \xrightarrow{\text{Hu}} & H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z}) & \xrightarrow{\varphi} & \underbrace{K_3\mathbb{Z} \otimes \mathbb{Z}/2}_{\cong \mathbb{Z}/2} & \rightarrow & K_4\mathbb{Z} \xrightarrow{\text{Hu}} H_4(B\text{St}(\mathbb{Z})^+; \mathbb{Z}) \longrightarrow 0 \\
 & & \downarrow f_* & & \downarrow \psi & & \\
 K_5\mathbb{F}_3 & \xrightarrow{\text{Hu}} & \underbrace{H_5(B\text{SL}(\mathbb{F}_3)^+; \mathbb{Z})}_{\cong \mathbb{Z}/2 \oplus \mathbb{Z}/13} & \xrightarrow{\chi} & \underbrace{K_3\mathbb{F}_3 \otimes \mathbb{Z}/2}_{\cong \mathbb{Z}/2} & \rightarrow & K_4\mathbb{F}_3 = 0
 \end{array}$$

Note that the Whitehead exact sequence can also be obtained from the Serre spectral sequence of the fibration $A(\mathbb{Z}) \rightarrow B\text{St}(\mathbb{Z})^+ \xrightarrow{p} K(K_3\mathbb{Z}, 3)$ (respectively $A(\mathbb{F}_3) \rightarrow B\text{SL}(\mathbb{F}_3)^+ \rightarrow K(K_3\mathbb{F}_3, 3)$), where p is the Postnikov approximation map and $A(\mathbb{Z})$ the fibre of p .

The homomorphism ψ is actually $f_* \otimes 1$ and, since $f_* : K_3\mathbb{Z} \cong \mathbb{Z}/48 \rightarrow K_3\mathbb{F}_3 \cong \mathbb{Z}/8$ is surjective (cf. [1, §3]), ψ is an isomorphism. Proposition 3.6 says that $\chi \circ f_*$ is surjective and, by commutativity of the diagram, that φ is surjective. The proof is then complete because the group $\text{St}(\mathbb{Z})$ and the space $B\text{St}(\mathbb{Z})^+$ have the same homology. \square

Proof of Theorem 1.5. The above diagram and Proposition 3.13 show that $\varphi(T) = 0$ (T denotes the torsion subgroup of $H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z})$). Therefore we have an exact sequence

$$K_5\mathbb{Z}/\text{torsion} \xrightarrow{\text{Hu}'} H_5(B\text{St}(\mathbb{Z})^+; \mathbb{Z})/\text{torsion} \xrightarrow{\varphi'} \mathbb{Z}/2$$

where Hu' (respectively φ') is induced by Hu (respectively φ). It follows from the surjectivity of φ that φ' is also surjective. Consequently Hu' is multiplication by 2. The analogous statement for the space $BSL(\mathbb{Z})^+$ is then a consequence of Remark 2.12. \square

References

- [1] D. Arlettaz, Chern-Klassen von ganzzahligen und rationalen Darstellungen diskreter Gruppen, *Math. Z.* 187 (1984) 49–60.
- [2] D. Arlettaz, On the homology and cohomology of congruence subgroups, *J. Pure Appl. Algebra* 44 (1987) 3–12.
- [3] P. Deligne and D. Sullivan, Fibrés vectoriels complexes à groupe structural discret, *C.R. Acad. Sci. Paris Sér. A* 281 (1975) 1081–1083.
- [4] J. Huebschmann, The cohomology of $F\Psi^q$, the additive structure, *J. Pure Appl. Algebra* 45 (1987) 73–91.
- [5] D. Quillen, On the cohomology and K -theory of the general linear groups over a finite field, *Ann. of Math.* 96 (1972) 552–586.
- [6] C. Soulé, Classes de torsion dans la cohomologie des groupes arithmétiques, *C.R. Acad. Sci. Paris Sér. A* 284 (1977) 1009–1011.
- [7] G.W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics 61 (Springer, Berlin, 1978).
- [8] J.H.C. Whitehead, A certain exact sequence, *Ann. of Math.* 52 (1950) 51–110.